# *Multiple Integrals Over Infinite Fields, and the Fourier Multiple Integral.**

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INTRODUCTION.

1. Multiple integrals over infinite fields have been defined in recent years in different ways by Pierpont,† Vallée-Poussin, ‡ and Hardy. § The integrals of Pierpont and of Vallée-Poussin exist only when they exist absolutely, but this is not the case with Hardy's integral. It is shown here that, for functions whose integrals over finite fields exist, these three infinite integrals are equivalent when any one of them exists absolutely. This paper is divided into three parts. In the first part (§§ 2–5) are developed relations between Hardy's integral and the iterated integrals, certain new theorems on inversion of the order of integration, and some fundamental properties of multiple infinite integrals, most of which are known for the case of integrals over finite fields but not for the infinite case. Among these last is the second theorem of the mean. Heretofore, this has not been proved, apparently, in the infinite case even for functions of one variable. The second part (§§ 6–8) deals with the behavior of integrals with respect to parameters. I am mainly concerned here with the infinite integrals, but it is necessary incidentally to extend to the case of integrals over finite fields in space of several dimensions theorems which are known only for integrals over one-dimensional fields. To save space, and because novel methods would not be introduced, the proofs of many of the statements in this part are not given. They have to do, however, with fundamental matters, including what are, in one dimension, well-known tests for termwise integration and differentiation; in the infinite case they include the so-called fundamental theorem of the integral calculus. A résumé of the third part (§§ 9–11), which contains certain applications to Fourier's double and quadruple integrals, is given at the beginning of § 9.

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* Read in part before the American Mathematical Society, February 27, 1915, April 29, 1916.

† *Transactions of the American Mathematical Society*, Vol. VII (1906), p. 169.

‡ *Cours d'Analyse*, 2d ed., Vol. II, pp. 69, 70, 108, 109.

§ *Messenger of Mathematics*, Vol. XXXII (1902–3), pp. 92 ff.

FUNDAMENTAL THEORY, §§ 2–5.

2. **Elementary Theorems.** As we shall not be concerned usually with integration over finite fields, we make the assumption always that the functions used are *absolutely* $L$-integrable * in each *limited* measurable set of their domain of definition. Let $R$ denote the infinite rectangle † where $x, y \geqq 0$, and $R_{\alpha,\beta}$ the finite rectangle where $0 \leqq x \leqq \alpha$, $0 \leqq y \leqq \beta$. Let $f(x, y)$ be defined in $R$. We may then define the following integrals, when the limits in question exist:

$$J_1(f) = \lim_{\alpha,\beta=\infty} \int_{R_{\alpha,\beta}} f\,dx\,dy,$$

$$J_2(f) = \lim_{\alpha=\infty} \lim_{\beta=\infty} \int_{R_{\alpha,\beta}} f\,dx\,dy,$$

$$J_3(f) = \lim_{\beta=\infty} \lim_{\alpha=\infty} \int_{R_{\alpha,\beta}} f\,dx\,dy,$$

$$J_4(f) = \lim_{\alpha=\infty} \int_0^\alpha dx \lim_{\beta=\infty} \int_0^\beta f\,dy = \int_0^\infty dx \int_0^\infty f\,dy,$$

$$J_5(f) = \lim_{\beta=\infty} \int_0^\beta dy \lim_{\alpha=\infty} \int_0^\alpha f\,dx = \int_0^\infty dy \int_0^\infty f\,dx.$$

The integral $J_1(f)$ is Hardy's definition referred to in the introduction, $J_4$ and $J_5$ are the usual iterated integrals, $J_2$ and $J_3$ are introduced chiefly for convenience in developing the theory of $J_1$, $J_4$ and $J_5$. When it is desired to integrate a function over a larger portion or over all of the infinite two-dimensional field, we let $R_1$ denote the rectangle $(x, y \geqq 0)$, $R_2$ the rectangle $(x < 0 \leqq y)$, $R_3$ the rectangle $(x, y < 0)$, and $R_4$ the rectangle $(y < 0 \leqq x)$, and let $R = R_1 + R_2 + R_3 + R_4$, and then define the integral over $R$ as the sum of the integrals over $R_1$, $R_2$, $R_3$, and $R_4$. These definitions could be generalized slightly by using the concept of the Harnack-Lebesgue integral, but that is not worth while for my present purpose. By $\lim_{\alpha,\beta=\infty}$ is meant the double limit ‡ as $\alpha$, $\beta$ diverge to plus infinity passing through all positive values. Most of the theorems, though stated only for two dimensions, may obviously be extended to $n$ dimensions.

The two following theorems are almost immediate consequences of our definition, and the succeeding lemma is known.

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* *I. e.*, integrable in the sense of Lebesgue.

† The case of $n$ variables is not essentially different from the case of two, but the treatment would be cumbersome. In § 6, however, where finite fields only are employed, and there is no loss of simplicity, $n$ variables are used.

‡ *E. g.*, Pierpont, *Theory of Functions of Real Variables*, Vol. I (1905), p. 195.

THEOREM 1.* *A necessary and sufficient condition that $J_1(f)$ shall exist is that, for every pair of constants $h, k>0$,*

$$\lim_{\alpha, \beta=\infty} \int_r f dx dy = 0,$$

where $r = R_{\alpha+h, \beta+k} - R_{\alpha, \beta}$.

THEOREM 2. *If $J_i(f)$ and $J_i(g)$ exist, and $c$ is a constant, $J_i(f)+J_i(g)=J_i(f+g)$, and $J_i(cf)=cJ_i(f)$.*

LEMMA. *If for all values of $\Delta\alpha, \Delta\beta \geqq 0$, $\phi(\alpha+\Delta\alpha, \beta+\Delta\beta) \geqq \phi(\alpha, \beta)$,* and *$\phi(\alpha, \beta)$ is less than a fixed number $M$, then $\lim_{\alpha, \beta} \phi(\alpha, \beta)$, $\lim_\alpha \lim_\beta \phi(\alpha, \beta)$, $\lim_\beta \lim_\alpha \phi(\alpha, \beta)$ all exist and are equal and not greater than $M$.*

THEOREM 3. *If, in general,† $0 \leqq f(x, y) \leqq g(x, y)$, and if $J_i(g)$ exists, so does $J_i(f)$, and $J_i(f) \leqq J_i(g)$.*

If $i=1$, 2, or 3, this is an immediate consequence of the lemma. If $i=4$, by hypothesis there exists, except perhaps for a null set, a positive $L$-integrable function,

$$L(x) = \int_0^\infty g dy,$$

whose integral from 0 to $\alpha$ is a monotonic increasing function of $\alpha$ and has the limit $J_4(g)$. Except perhaps for a null set of $x$'s,

$$\int_0^\beta f dy = \leqq \int_0^\beta g dy,$$

and the former is a monotonic increasing function of $\beta$ whose limit $l(x)$ is less than or at most equal to $L(x)$. The integral of $l(x)$ from 0 to $\alpha$ is a monotonic increasing function of $\alpha$ and hence

$$\lim_\alpha \int_0^\alpha l(x) dx = J_4(f) \leqq \lim_\alpha \int_0^\alpha L(x) dx = J_4(g).$$

COROLLARY.‡ *The function $J_i(f)$ exists if $J_i(g)$ does, when $f(x,y)=g(x,y)$ in a part of $R$ and zero elsewhere, provided that the field is so divided that $f$ is L-integrable over every limited, measurable set in each part.*

THEOREM 4. *A necessary and sufficient condition that $J_i(f)$ shall exist is that there shall exist two other functions $g(x, y)$, $G(x, y)$, for both of*

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* The proof of this depends on the following theorem, well known for functions of *one* variable: A necessary and sufficient condition that the double limit of $\phi(\alpha, \beta)$ shall exist, as $\alpha, \beta$ become infinite, is that, for every pair of constants $h, k>0$, $\lim_{\alpha, \beta=\infty} [\phi(\alpha+h, \beta+k) - \phi(\alpha, \beta)] = 0$.

† *I. e.*, except perhaps at a set of points of measure zero. Such a set will be called a null set.

‡ This is not necessarily true if $g$ has both signs. Indeed, this property, together with the properties in Theorem 2, would be sufficient to show that $|f|$ would be integrable, no matter what the definition of integration for $f$.

*which $J_i$ exists, such that in general $g(x, y) < f(x, y) < G(x, y)$. Then also $J_i(g) \leqq J_i(f) \leqq J_i(g)$.*

Since $0 < f - g < G - g$, $J_i(f-g) \leqq J_i(G-g)$. Then, since $J_i(g)$ exists, we have $J_i(f) \leqq J_i(G)$.

COROLLARY 1. *The same is true, with an appropriate change in the last statement, if $g \leqq f \leqq G$.*

COROLLARY 2. *The function $J_i(f)$ exists if it exists absolutely, or if $f$ is the product of two positive functions which are less than unity and for each of which $J_i$ exists, or if it is the p-th power $(p > 1)$ of such a function.*

COROLLARY 3. *Under the usual conditions the mean-value theorem, $mJ_i(g) \leqq J_i(fg) \leqq MJ_i(g)$, is valid.*

COROLLARY 4. *If $J_i(f^2)$ and $J_i(g^2)$ exist, then $J_i(fg)$ exists absolutely, and the inequality of Schwarz, $J_i^2(fg) \leqq J_i(f^2) \cdot J_i(g^2)$, is valid.**

CASE I: $(i=1)$. Let $\psi = f$ and $\phi = 0$, $\gamma = g$ and $\delta = 0$ where $fg \geqq 0$, and let $\phi = f$ and $\psi = 0$, $\delta = g$ and $\gamma = 0$ where $fg < 0$. The same equalities hold when both sides are squared, and by the corollary of Theorem 3 the following exist: $J_1(\psi^2)$, $J_1(\phi^2)$, $J_1(\gamma^2)$, $J_1(\delta^2)$. By the inequality for finite fields,

$$\left(\int_{R_{\alpha,\beta}} \psi\gamma\, dxdy\right)^2 \leqq \left(\int_{R_{\alpha,\beta}} \psi^2 dxdy\right)\left(\int_{R_{\alpha,\beta}} \gamma^2 dxdy\right) \leqq J_1(\psi^2) \cdot J_1(\gamma^2).$$

Since $\psi\gamma \geqq 0$, the function of $\alpha$, $\beta$ on the left satisfies the conditions of the lemma to Theorem 3. Therefore $J_1^2(\psi\gamma)$ exists, and so does $J_1(\psi\gamma)$. Similarly, $J_1(\phi\delta)$ exists, and since by the early equalities $|fg| = \psi\gamma - \phi\delta$, so does $J_1(|fg|)$. Now, proceeding to the limit in the relation,

$$\left(\int_{R_{\alpha,\beta}} fg\, dxdy\right)^2 \leqq \left(\int_{R_{\alpha,\beta}} f^2 dxdy\right)\left(\int_{R_{\alpha,\beta}} g^2 dxdy\right),$$

we may establish the proposition.

CASE II: $(i > 1)$. This may be referred to Case I by the use of the corollary to Theorem 5 and the first part of Corollary 2, Theorem 4.

3. **The Integrals of Pierpont and Vallée-Poussin.**† In considering these definitions we shall assume as in § 2 that $f(x, y)$ is absolutely $L$-integrable in every limited measurable part of the field $R$. This is not done by Pierpont, but it is better for our present purpose to dispense with the extra element of generality which he obtains by omitting this condition. Let $R$ be defined as before, and let $D_1 < D_2 < \ldots.$ be any infinite sequence of limited, measurable

*It does not follow, as in the finite case, that $f$ is integrable if $f^2$ is.

†Cf. § 1.

point sets of such a nature that each $D$ is wholly in $R$ and that every point of $R$ is in some $D$.

*Vallée-Poussin's Definition,* $V(f)$. Let $f \geqq 0$. If the following limit exists and is independent of the choice of $D$'s,

$$V(f)=\int_R f dxdy=\lim_{i=\infty}\int_{D_i} f dxdy. \tag{1}$$

Let $f$ be unrestricted as to sign. Set $f=f_1-f_2$, where $f_1, f_2 \geqq 0$. When the integrals on the right exist,

$$V(f)=\int_R f dxdy=\int_R f_1 dxdy-\int_R f_2 dxdy. \tag{2}$$

*Pierpont's Definition,* $P(f)$. Let $f$ be unrestricted as to sign. Let the following be an infinite sequence of limited measurable sets, $E_1, E_2, \ldots\ldots$; and let $E_i$ contain every point whose distance from the origin is less than $i$; it may contain other points. When the following limit exists and is independent of the choice of $E$'s,

$$P(f)=\int_R f dxdy=\lim_{i=\infty}\int_{E_i} f dxdy. \tag{3}$$

It will be shown that this definition is equivalent to the definition:

$$P(f)=\lim_{i=\infty}\int_{D_i} f dxdy. \tag{4}$$

It is obvious that if (3) exists (4) exists and gives the same result. To prove the converse we let (4) exist. Suppose that, for a certain $E$ set, the limit in (3) should not exist, *i. e.*, suppose that the sequence

$$\int_{E_1} f dxdy, \quad \int_{E_2} f dxdy, \ldots\ldots \tag{5}$$

should diverge. In this sequence there would exist a divergent sub-sequence

$$\int_{E'} f dxdy, \quad \int_{E''} f dxdy, \ldots\ldots \tag{6}$$

for which $E'<E''<\ldots\ldots$ For example, we might take $E'=E_1$, and then, since $E_1$ is limited, there would exist an $E_i$ which would include all points as near to the origin as any of those of $E_1$; let $E''=E_i$, or some $E$ farther on in the sequence, if necessary, in order to make (6) diverge. This sub-sequence would be a $D$ sequence.

Lemma. *If $f(x, y) \geqq 0$ and the following limit exists for one set of $D$'s, it does for every set:*

$$\lim_{i=\infty}\int_{D_i} f dxdy.$$

THEOREM 5. (*a*) *If* $f \geqq 0$, $V(f) = P(f) = J_1(f)$, *provided one of these exists.* (*b*) *If* $V(f)$ *or* $P(f)$ *exists, it exists absolutely, and then* $V(f) = P(f) = J_1(f)$. (*c*) $J_1(f)$ *may exist when* $V(f)$ *and* $P(f)$ *do not.*

(*a*) Since $R_{a,\beta}$ is a special case of $E_i$, if $P$ exists, so does $J_1$, and $J_1 = P$. By the lemma, if $J_1$ exists, so does $V$, and $V = J_1$. By definition $V = P$.

(*b*) It is known and obvious that if $V(f)$ exists so does $V(|f|)$, and

$$V(|f|) = \lim_{i=\infty} \int_{D_i} f_1 dxdy + \lim_{i=\infty} \int_{D_i} f_2 dxdy, \quad V(f) = \lim_{i=\infty} \int_{D_i} f_1 dxdy - \lim_{i=\infty} \int_{D_i} f_2 dxdy,$$

where $|f| = f_1 + f_2$, $f = f_1 - f_2$. By (*a*), then, $J_1(f_1) = V(f_1)$, $J_1(f_2) = V(f_2)$. Therefore, $J_1(f) = J_1(f_1) - J_1(f_2) = V(f)$. A similar result may be obtained for $P(f)$, since it may be shown that this integral also obeys Theorem 2. Thus (*b*) is proved if the hypothesis be that $V(f)$ exists. It only remains to show that if $P(f)$ exists so does $V(f)$. The following equation is true if two of the limits exist:

$$P(f) = \lim_{i=\infty} \int_{D_i} f dxdy = \lim_{i=\infty} \int_{D_i} f_1 dxdy - \lim_{i=\infty} \int_{D_i} f_2 dxdy.$$

If it is true, $V(f)$ exists. If it is not true, but $P(f)$ exists; since $f_1, f_2 \geqq 0$, both of the last two integrals must diverge to plus infinity for some set of $D$'s, and therefore by the lemma for all sets. On this hypothesis we will show that there exists a set of $D$'s for which the first limit does not exist. Let $R_1$ be a circle of unit radius about the origin. Let

$$k_1 = \int_{R_1} f dxdy.$$

Since the integral of $f_1$ over $R_1$ is not infinite, and since by definition $f_1 = f$ where $f \geqq 0$, there must exist outside of $R_1$ a limited set $A_1$, in which $f \geqq 0$, such that

$$\int_{A_1} f_1 dxdy > 1 - k_1.$$

Hence,

$$\int_{R_1 + A_1} f dxdy = \int_{R_1} f dxdy + \int_{A_1} f dxdy > k_1 + 1 - k_1 = 1.$$

Let $D_1 = R_1 + A_1$. Let $R_2$ be a circle about the origin including all of $D_1$. Let

$$k_2 = \int_{R_2} f dxdy.$$

The integral of $f_1$ over $R_2$ is not infinite, and therefore, as before, there exists a set $A_2$ outside of $R_2$ where $f \geqq 0$ such that

$$\int_{R_2 + A_2} f dxdy = \int_{R_2} f dxdy + \int_{A_2} f dxdy = \int_{R_2} f dxdy + \int_{A_2} f_1 dxdy > 2.$$

Let $D_2 = R_2 + A_2$; etc.

($c$) Hardy * has given some interesting examples of this. One which will concern us later is $J_1(\cos xy) = \pi/2$.

COROLLARY.† *If $J_i(|f|)$ exists, so does $J_j(|f|)$, and $J_i(|f|) = J_j(|f|)$, and $J_i(f) = J_j(f)$.*

Vallée-Poussin has shown that if $V(|f|)$ exists, $J_4(|f|) = J_5(|f|) = V(|f|)$, and that $J_4(f) = J_5(f) = V(f)$. By the theorem, then, the corollary is proved for the cases $i, j = 1, 4, 5$. The remaining cases may easily be proved by the use of the lemma to Theorem 3.

4. **More Existence and Inversion Theorems.** THEOREM 6. *Let $J_i(f)$ exist. A necessary and sufficient condition that $J_j(f)$ shall exist is that there shall exist a function $g(x, y)$ which is in general less than $f(x, y)$ for which both $J$'s exist.*

Necessity. Let $q(x, y) > 0$, and let $J_j(q)$ exist. We may take $g = f - q$, for, since $J_j(f)$ and $J_j(q)$ exist, so does $J_j(f - q)$.

Sufficiency. Since $J_i(f-g)$ exists and $f - g \geqq 0$, $J_j(f-g)$ exists by the preceding corollary. Since now $J_j(g)$ exists, so does $J_j(f - g + g)$.

COROLLARY 1. *Let $J_i(f)$ exist. A necessary and sufficient condition that $J_j(f) = J_i(f)$ is that there shall exist a function $g(x, y)$, in general less than $f(x, y)$, for which this is true.*

$J_i(f-g) = J_j(f-g) = J_j(f) - J_j(g)$. Add the equality, $J_i(g) = J_j(g)$.

COROLLARY 2. *Let one of the iterated integrals $J_4(f)$, $J_5(f)$ exist. A necessary and sufficient condition that inversion of the order of iteration be allowable is that there shall exist a function $g(x, y)$, in general less than $f(x, y)$, for which the inversion is allowable.*

COROLLARY 3.‡ *A necessary and sufficient condition that both iterated integrals of $f$ shall exist and be equal is that this shall be true of two functions $g(x, y)$, $G(x, y)$ for which in general $g < f < G$.*

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* *Loc. cit.*, § 1, pp. 159 ff.

† Vallée-Poussin, *loc. cit.*, Vol. II, p. 121. Bromwich (*Proceedings of the London Mathematical Society*, Vol. I (1903), p. 181, third footnote) indicates that this result follows from the lemma to Theorem 3, but really only the last part, in which it is said that $J_1 = J_2 = J_3$, follows from that lemma. Indeed, if it did follow also that $J_1 = J_4 = J_5$, as he states, his result would antedate its discovery for the finite case by four years (see also Hobson, same *Proceedings*, Vol. VIII (1910), p. 22).

I am omitting from this paper the subject of transformation of variables in these integrals. It is true, however, that, if any $J_i$ exists absolutely, it may be transformed into an equivalent double, or iterated, integral over the unit square. Another proof of the corollary may be devised by the use of transformations of this sort.

‡ Cf. W. H. Young, same *Proceedings*, Vol. IX (1910–11), p. 323. Young's theorem gives only sufficient criteria that inversion of the order of iteration be permissible, and involves the additional hypothesis that this inversion be permissible for integrals of $g$ and $G$ over one-way infinite fields.

COROLLARY 4.* *(a) Inversion of the order of iteration of $f$ over the one-way infinite field $R_\alpha(0 \leqq x \leqq \alpha, 0 \leqq y)$ is allowable if one of the iterated integrals of $|f|$ exists. (b) If one of the iterated integrals of $f$ over $R_\alpha$ exists, a necessary and sufficient condition that inversion be allowable is that there shall exist a function $g(x, y)$, in general less than $f(x, y)$, for which it is allowable. (c) A necessary and sufficient condition that both iterated integrals over $R_\alpha$ shall exist and be equal is that there shall exist two functions $g(x, y)$, $G(x, y)$ for which this is so and for which in general $g<f<G$.*

COROLLARY 5. *The theorem and previous corollaries are true if we write, instead of $g<f(f<G)$, $g \leqq f(f \leqq G)$, or if, in the theorem and first two corollaries, we substitute $G>f$ for $g<f$.*

THEOREM 7. *(a) If $J_1(f)$ exists, and $J_2(f)$ exists over $R_\alpha$ for sufficiently large values of $\alpha$ (say $\alpha>C$) then $J_2(f)=J_1(f)$; similar statements may be made for $J_3$ and the one-way infinite field $R_\beta(0 \leqq y \leqq \beta, 0 \leqq x)$. (b) Let inversion† of the order of iteration over $R_\alpha(\alpha>C)$ be allowable; then $J_2(f)=J_4(f)$ if either exists; and similar statements may be made for $J_3$ and $J_5$ and $R_\beta$.*

COROLLARY. *Sufficient conditions that all the $J$'s shall exist and be equal are that inversion be allowable over both $R_\alpha$ and $R_\beta(\alpha, \beta>C)$, and that $J_1$ and one of each of the pairs $(J_2, J_4)$, $(J_3, J_5)$ shall exist. Then also all the $J$'s are equal.*

The proofs of the theorems and corollaries of this section are simple.

5. **Second Theorem of the Mean.**‡ THEOREM 8. *Let $f(x, y)$ be limited in $R$ and L-integrable over every finite measurable set, and let $g(x, y)$ be absolutely L-integrable in $R$. Then, if $l \leqq f(x, y) \leqq L$, there exists a $k$ such that*

$$J=\int_R f(x, y)\, g(x, y)\, dxdy=l\int_{R-A_k} g\,dxdy+L\int_{A_k} g\,dxdy;$$

*where, if $f$ has the property that the set of points $E_k$, where $f=k$, is always null, $A_k$ is defined as the set where $f>k$; and otherwise $A_k$ is the set where*

* Parts (*a*) and (*c*) are known. See also note to Theorem 7 (*b*). By the existence of the integral of $f$ over $R_\alpha$ we mean the existence over $R$ of the integral of the function which equals $f$ in $R_\alpha$ and zero elsewhere.

† For criteria for this condition we have, not only Theorem 6, Corollary 4, but also the usual condition that, assuming $|f|$ $L$-integrable over finite fields, termwise integration with respect to $x$ of the $\beta$ sequence—viz: the integral of $f$ with respect to $y$ from 0 to $\beta$—be allowable in $(0, \alpha)$. This assumes implicitly the existence of one iterated integral.

‡ Proved for the case where $R$ is finite by Lebesgue, *Annales de l'École Normale*, (3) Vol. XXVII (1910), p. 444; proved in a slightly more general form by the author, *Mathematische Annalen*, Vol. LXXV (1914), p. 285. There will be used in this section much of the notation and some of the results of the latter paper.

*$f>k$, plus perhaps certain points where $f=k$. In the second case the set $A_k$ may be defined by the use of an associated function $q(x, y)$.*

CASE 1: ($l=0$, $\widehat{E}_k=0$).* Consider a monotonic sequence of circles whose union † is $R: R_1<R_2<\ldots\ldots$ Let $A_k$ be the set of points where $f>k$. Let $A'_{k_i}$ refer to the parts of $A_{k_i}$ in $R_i$. By the finite case there exists for each $R_i$ a $k_i$ such that

$$\int_{R_i} fg = L\int_{A'_{k_i}} g. \tag{1}$$

By Theorem 4, $fg$ is absolutely $L$-integrable in $R$, and by Theorem 5

$$J=\int_R fg=\lim_{i=\infty}\int_{R_i} fg=L\lim_{i=\infty}\int_{A'_{k_i}} g; \tag{2}$$

so the last limit exists. It only remains to show that there is a definite $k$ and $A_k$ for which

$$\lim_{i=\infty}\int_{A'_{k_i}} g=\int_{A_k} g. \tag{3}$$

The left-hand side may be considered as the limit of any infinite sub-sequence selected from the sequence

$$\int_{A'_{k_1}} g,\quad \int_{A'_{k_2}} g,\ \ldots\ldots \tag{4}$$

Suppose the sub-sequence to be the terms corresponding to $i=a_1<a_2<\ldots.$, where these numbers are so chosen that, if $k$ be taken as the lower limit of the sequence $k_1, k_2, \ldots.$, it is also the limit of either (5) or (6),

$$k_{a_1}\geqq k_{a_2}\geqq\ldots., \tag{5}$$

$$k_{a_1}\leqq k_{a_2}\leqq\ldots.. \tag{6}$$

To be assured that this is possible we first note that whenever the points $k_1, k_2, \ldots.$ constitute an infinite point set, there exists a lower limit, for the set is limited ($0\leqq k_i\leqq L$). Whenever the points do not constitute an infinite point set, certain of these numbers are repeated an infinite number of times; let $k$ be the least of these numbers. It is immaterial to the proof whether (5) or (6) holds. Assume (5); and to avoid the additional subscripts let it be assumed that (4) *is* the sequence corresponding to (5), *i. e.*, that $k$ is the limit of $k_i$, and that

$$k_1\geqq k_2\geqq\ldots.. \tag{7}$$

Then

$$A'_{k_1}\leqq A'_{k_2}\leqq\ldots.. \tag{8}$$

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* By $\widehat{E}$ is meant the measure of $E$, a concept which may be defined in the infinite case as the integral over $E$ of unity.

† By "union" of $(R_i)$ is meant the totality of points each of which belongs to some $R_i$.

The limit of this sequence may be an infinite field. For $\varepsilon > 0$ prescribed, let $R_a$ be so large that

$$\int_{R-R_a} |g| < \varepsilon. \tag{9}$$

Let

$$A''_{k_1} \leqq A''_{k_2} \leqq \ldots . \tag{10}$$

refer to the portions of $A'_{k_1}$, etc., in $R_a$. If $i < \alpha$, $A''_{k_i}$ may not refer to all of the points of $R_a$ where $f > k_i$, for $A'_{k_i}$ may not include them all; but if $i \geqq \alpha$, $A''_{k_i}$ does include all those points, and only those. $A''_k$ is the set in $R_a$ where $f > k$; and so, by the reasoning of my *Annalen* paper,* $A''_k$ is the union of $A''_{k_i}$ and $\lim_{i=\infty} \widehat{A}''_{k_i} = \widehat{A}''_k$. Thus, on account of the absolute continuity of the integral of $g$ in $R_a$, there exists an $i_\epsilon > \alpha$ and so large that if $i > i_\epsilon$ $(\widehat{A}''_k - \widehat{A}''_{k_i})$ is sufficiently small to make

$$\int_{A''_k - A''_{k_i}} |g| < \varepsilon. \tag{11}$$

Now the set $(A_k - A'_{k_i})$ is made up of two parts, $(A''_k - A''_{k_i})$ and $B$, where $B$ is some set outside of $R_a$. Therefore, by (9) and (11),

$$\left| \int_{A_k - A'_{k_i}} g \right| < \int_{A''_k - A''_{k_i}} |g| + \int_B |g| < 2\varepsilon,$$

which proves (3), and hence the proposition.

CASE 2: ($l = 0$, $\widehat{E}_k \geqq 0$). The only new difficulty here is with inequality (11). It is obvious that $(\widehat{A}''_k - \widehat{A}''_{k_i})$ may not become small at pleasure because of the non-null sets that may be included. Therefore a method must be found of dividing up these non-null $E_k$ sets. The measures of some of them may be infinite. We first note that there are only an enumerable number of values of $k$, say $k_1, k_2, \ldots$, where $\widehat{E}_k > 0$. This follows from the fact that, if $A_k$ be, as before, the set where $f > k$, $\widehat{A}_k$ is a monotonic decreasing function of $k$, and $k_i$ is a point of discontinuity of $\widehat{A}_k$. It is known that such points are at most enumerably infinite for monotonic functions. We now define, corresponding to the parameter $\lambda (0 \leqq \lambda \leqq L+1)$, a family of point sets $\Omega_\lambda$, and a limited function $q(x, y)$ which is associated in a certain way with $f(x, y)$, and which is constant at most in null sets, and has the property that the set of points where $q > \lambda$ is $\Omega_\lambda$. Let $c_i$ be any positive number such that

$$c_1 + c_2 + \ldots = 1. \tag{12}$$

* *Loc. cit.*, p. 275, Lemma 2.

Let $q(x, y)$ be defined as the limit of the monotonic sequence,

$$f(x, y) \leqq q_1(x, y) \leqq q_2(x, y) \leqq \dots \leqq f(x, y)+1, \tag{13}$$

where $q_1, q_2, \dots$ are defined as follows: at points where $f<k_1$, let $q_1$ equal $f$; at points where $f=k_1$, let $q_1$ equal $c_1 e^{-x^2-y^2}+f$; and at points where $f>k_1$, let $q_1$ equal $f+c_1$. Here the function $c_1 e^{-x^2-y^2}+k_1$ has the properties that it is constant only in null sets, that it lies between $k_1$ and $k_1+c_1$, and that the set of points where it is greater than any number between these limits is a circle. Other definitions of $q_1$ might obviously be chosen, resulting in other associated functions $q$. Next, let $q_2$ be equal to $q_1$ where $f<k_2$, equal to $c_2 e^{-x^2-y^2}+q_1$ where $f=k_2$, and equal to $q_1+c_2$ where $f>k_2$, etc. Let $\Omega_\lambda$ be the set of points where $q(x, y)>\lambda \, (0 \leqq \lambda \leqq L+1)$. The proof that $q$ is constant at most in null sets is omitted because the fact seems obvious.

It will now be shown that each $\Omega_\lambda$ is divisible into two parts; one is part or all of the set where $f$ equals a certain corresponding $k_\lambda$, and the other is the whole of the set where $f>k_\lambda$. Consider the set $A_k$ where $f>k$, and suppose the $k_i$'s (7) which are not greater than $k$ to be denoted by $\alpha_i$'s and their corresponding $c_i$'s by $s_i$'s, thus:

$$\alpha_1, \alpha_2, \dots \leqq k; \; s_1, s_2, \dots .$$

In $A_k$,

$$q \geqq f+s_1+s_2+\dots > k+s_1+s_2+\dots . \tag{14}$$

For example, if $k_1=\alpha_1$, $k_2>k$, $k_3=\alpha_2, \dots$; in $A_k$, $q_1=f+s_1$, $q_2 \geqq q_1=f+s_1$, $q_3=q_2+s_2 \geqq f+s_1+s_2$, etc., and finally $q_1 \leqq q_2 \leqq \dots \leqq q$. In $R-A_k$,

$$q \leqq f+s_1+s_2+\dots \leqq k+s_1+s_2+\dots , \tag{15}$$

for here, for every $j$, $q_j \leqq f+s_1+s_2+\dots$, and finally $q=\lim_j q_j$. Thus, if $\lambda=k+s_1+s_2+\dots$, $q>\lambda$ in $A_k$ and $q \leqq \lambda$ in $R-A_k$, and so in this case $\Omega_\lambda=A_k$. If now it could be shown that to every $\lambda$ there corresponds a $k$ such that $\lambda=k+s_1+s_2+\dots$, the part of our proposition considered in this paragraph would be established, but this is not always true. It can, however, be proved except for an enumerable number of intervals where

$$\lambda_{k_i}-c_i<\lambda \leqq \lambda_{k_i} . \tag{16}$$

For $s_1+s_2+\dots$ is a monotonic increasing function of $k$, and therefore, considered as a function of $k$, $\lambda=k+s_1+s_2+\dots$ is always increasing, and is defined for each $k$ in $(0, L)$. Moreover, it is always continuous on the right, for, for any $\bar{k}$ which is not a $k_i$, there exists a $\delta$ so that the interval $(k, k+\delta)$ does not contain any of the points $k_1, k_2, \dots, k_n$ ($n$ finite). Therefore,

$$|\lambda(\bar{k}+\delta)-\lambda(\bar{k})|<|\delta|+\sum_n^\infty c_i<\varepsilon,$$

if $n$ and $\delta$ are chosen properly. This is true for $\delta>0$ even if $\bar{k}$ is a $k_i$, but in this case every interval $(\bar{k}, \bar{k}+\delta)$, $\delta<0$, would include $k_i$, and hence at such points $\lambda$ would have a discontinuity on the left equal to $c_i$. The correspondence desired is proved. We proceed to consider values of $\lambda$ of (16). First, all points of $A_{k_i}$ are in $\Omega_\lambda$, for, if $f>k_i$, $q>\lambda_{k_i}=k_i+s_1+s_2+\ldots.(c_i=s_1)$, that is $q>\lambda$. Secondly, $\Omega_\lambda$ contains no other points, except some where $f=k_i$; for, if $P$ is a point where $f(P)<k_i$, let $f(P)=k_i-\gamma$. To this value of $k$ there is a corresponding $\bar{\lambda}=k_i-\gamma+\sigma_1+\sigma_2+\ldots.$, where $\sigma_1, \sigma_2, \ldots.$ are the $s_i$'s for $k_i-\gamma$; their sum is less than $s_1+s_2+\ldots.$ by at least $c_i=s_1$. By (15)

$$q(P)\leqq k_i-\gamma+\sigma_1+\sigma_2+\ldots.\leqq k_i-\gamma+s_1+s_2+\ldots.-s_1<\lambda_{k_i}-c_i<\lambda,$$

and therefore $P$ is not in $\Omega_\lambda$. Thus, in every case there is a $k_\lambda$ corresponding to $\lambda$ for which the conditions are satisfied.

It may now be shown that

$$J=L\int_{\Omega_\lambda} g.$$

As in Case 1 (1),

$$\int_{R_i} fg=L\int_{\Omega'_{\lambda_i}} g,$$

where $\Omega'_{\lambda_i}$ is the part of $\Omega_{\lambda_i}$ in $R_i$. For $q$ is for $R_i$ precisely an associated function in the sense of my paper for the case when $R$ is limited. The rest of the reasoning of Case 1 now holds, *mutatis mutandis.*

CASE 3: $(l\neq 0)$. We may obviously consider Case 1 as a special case of Case 2, where $q=f$. By Case 2,

$$\int_R [f(x, y)-l]g(x, y)=(L-l)\int_{\Omega_\lambda} g=L\int_{\Omega_\lambda} g-l\int_R g+l\int_{R-\Omega_\lambda} g.$$

Add

$$\int_R lg=l\int_R g.$$

COROLLARY. *If $f(x)$ is monotonic decreasing in $(a, \infty)$ and $l\leqq f(x)\leqq L$, and if $g(x)$ is absolutely L-integrable in the same interval, there exists a $\lambda(a\leqq\lambda\leqq\infty)$ such that*

$$\int_a^\infty fg=l\int_\lambda^\infty g+L\int_a^\lambda g.$$

It is to be noted also that the theorem gives even for one dimension a much more general statement than this corollary.

INTEGRALS CONTAINING PARAMETERS, §§ 6–8.

6. **Integrals over Finite Fields.*** As a part of my discussion of integrals containing parameters I intend to extend to integrals over infinite fields certain theorems of W. H. Young † on integrals of functions of one variable. It is necessary first to make the extension to $n$ dimensions for finite fields, and in this section I shall cite certain of these results. They may be deduced more simply from Vitali's test ‡ of equi-absolute continuity, than by extending to $n$ dimensions Young's idea of upper (lower) semi-continuity, but I will not give the proofs. In passing, however, I call attention to one theorem (11) which is not related to Young's, but which is a generalization of a very useful theorem on integrals not containing parameters.

THEOREM 9. *Let $A$ be a limited measurable point set in space of $n$ dimensions. Let $lim_{n,m} f_{n,m}(x, y, \ldots) = f(x, y, \ldots)$ in $A$, in general. The necessary and sufficient condition that*

$$\lim_{n,\, m=\infty} \int_A f_{nm} dA = \int_A f dA$$

*is that there shall exist two functions $g_{n,m}(x, y, \ldots)$, $G_{n,m}(x, y, \ldots)$ whose limits are $g(x, y, \ldots)$ and $G(x, y, \ldots)$, respectively, for which this is true, and such that, in general, $g_{n,m} < f_{n,m} < G_{n,m}$.*

COROLLARY 1. *An analogous theorem is true for iterated integrals.*

COROLLARY 2. *If $f_{n,m}(x, y, \ldots) \geqq 0$, and its limit is $f(x, y, \ldots)$, and if also the limit of its integral over $A$ is the integral of $f$ over $A$, then a similar statement is true for the limit of its integral over any measurable part of $A$.*

THEOREM 10. *If the limit of a monotonic sequence of integrable functions $0 \leqq f_1(x, y, \ldots) \leqq f_2(x, y, \ldots) \leqq \ldots$, is integrable in $A$, the sequence is integrable termwise.*

THEOREM 11.§ *If in $A$ $f(x, y, \ldots)$ is the limit of a function $f_{n,m}(x, y, \ldots)$ which is nowhere negative, and the limit of whose integral is zero, then in general $f = 0$.*

The proof is immediate if it can be shown that the integral of $f$ over $A$ exists, for in that case the integral of $f_{n,m}$ is obviously equi-absolutely con-

* It is not assumed in this section, as it is elsewhere, that the functions in question have integrals over finite measurable fields. The case of $n$ parameters is not essentially different from that of two.

† *Loc. cit.*, pp. 315 ff.

‡ *Rendiconti del Circolo di Palermo*, Vol. XXIII (1907) pp. 137 ff. Vitali enunciated his test only for one dimension, but his method is valid for $n$ dimensions.

§ This is a generalization of a well-known theorem. Cf. Pierpont, *The Theory of Functions of Real Variables*, Vol. II (1911), p. 385, § 402 (2).

tinuous and approaches the integral of $f$ as a limit. Thus the integral of $f$ is zero, and the theorem of which this is a generalization applies. Suppose the integral of $f$ over $A$ did not exist. We could then let $A_k$ be the set where $f > k$, and set $\phi_k$ equal to zero in $A_k$ and to $f$ in $(A - A_k)$. Then the integral of $\phi_k$ over $A$ would diverge to plus infinity as $k$ became infinite. This would be impossible, for suppose that for each $k$ $\phi_{n,m;k}$ equals 0 in $A_k$ and $f_{n,m}$ elsewhere. Then for each $k$ $\phi_{n,m;k}$ would satisfy the conditions of the theorem and $\phi_k$ would be integrable. Therefore the integral of $\phi_k$ over $A$ would equal zero for each $k$.

7. **Continuity.** LEMMA 1. *If* $0 \leqq f(x, y) \leqq g(x, y)$, *and* $J_i(g)$ *exists, then*

$$0 \leqq J_i(f) - \int_{R_{\alpha,\beta}} f dxdy \leqq J_i(g) - \int_{R_{\alpha,\beta}} g dxdy.$$

LEMMA 2. *If for each* $(n, m)$ *the function* $f_{n,m}(x, y)$ *satisfies the conditions imposed on* $f$ *in Lemma* 1, *and if* $f(x, y) = lim_{n,m} f_{n,m}(x, y)$ *in general, then* $J_i(f_{n,m})$ *exists and its limit is* $J_i(f)$.

Obviously $0 \leqq f \leqq g$, and so by Theorem 3 $J_i(f_{n,m})$ and $J_i(f)$ exist. For an arbitrary $\varepsilon > 0$, let $\alpha$, $\beta$ be chosen so large that

$$J_i(g) - \int_{R_{\alpha,\beta}} g < \varepsilon.$$

By Lemma 1, then,

$$0 \leqq J_i(f) \quad - \int_{R_{\alpha,\beta}} f \leqq J_i(g) \quad - \int_{R_{\alpha,\beta}} g < \varepsilon, \tag{1}$$

$$0 \leqq J_i(f_{n,m}) - \int_{R_{\alpha,\beta}} f_{n,m} \leqq J_i(g) - \int_{R_{\alpha,\beta}} g < \varepsilon. \tag{2}$$

By Theorem 9 there exists for these fixed values of $\alpha$, $\beta$ an $n_\epsilon$ so large that, if $n, m > n_\epsilon$,

$$\left| \int_{R_{\alpha,\beta}} f - \int_{R_{\alpha,\beta}} f_{n,m} \right| < \varepsilon. \tag{3}$$

From (1), (2), and (3),

$$|J_i(f) - J_i(f_{n,m})| < 3\varepsilon, \qquad (n, m > n_\epsilon).$$

LEMMA 3. *Except perhaps for a null set in* $R$, *let* $g_{n,m}(x, y) \geqq 0$, $lim_{n,m} g_{n,m}(x, y) = g(x, y)$, *and* $lim_{n,m} J_i(g_{n,m}) = J_i(g)$. *Then, for all values of* $\alpha$, $\beta$ *uniformly,*

$$\lim_{n,m} \int_{R_{\alpha,\beta}} g_{n,m} dxdy = \int_{R_{\alpha,\beta}} g dxdy.$$

Let $\phi_{n,m}=g_{n,m}$ and $\psi_{n,m}=g$ where $g_{n,m}\geqq g$, and let $\phi_{n,m}=g$ and $\psi_{n,m}=g_{n,m}$ where $g_{n,m}<g$. Then $0\leqq\psi_{n,m}<\phi_{n,m}$,

$$0\leqq\phi_{n,m}-\psi_{n,m}=|g_{n,m}-g|, \tag{1}$$

$\lim_{n,m}\phi_{n,m}=\lim_{n,m}\psi_{n,m}=g$, and $0\leqq\psi_{n,m}\leqq g$. By Lemma 2, then,

$$\lim_{n,m}J_i(\psi_{n,m})=J_i(g). \tag{2}$$

By definition, $\phi_{n,m}+\psi_{n,m}=g_{n,m}+g$, and hence $\lim_{n,m}J_i(\phi_{n,m}+\psi_{n,m})=2J_i(g)$. This, with (1) and (2), proves that $\lim_{n,m}J_i(|g_{n,m}-g|)=0$. Now, uniformly in $\alpha$, $\beta$,

$$\left|\int_{R_{\alpha,\beta}}(g_{n,m}-g)\right|\leqq\int_{R_{\alpha,\beta}}|g_{n,m}-g|\leqq J_1(|g_{n,m}-g|).$$

Lemma 4. *Except perhaps for a null set in $R$, let $0\leqq f_{n,m}(x,y)\leqq g_{n,m}(x,y)$, let $\lim_{n,m}f_{n,m}(x,y)=f(x,y)$, and $\lim_{n,m}g_{n,m}(x,y)=g(x,y)$, and $\lim_{n,m}J_i(g_{n,m})=J_i(g)$. Then $J_i(f_{n,m})$ exists and its limit is $J_i(f)$.*

Since obviously $0\leqq f\leqq g$, we have, by Lemma 1, for all values of $n$, $m$; $\alpha$, $\beta$,

$$0\leqq J_i(f)\quad-\int_{R_{\alpha,\beta}}f\quad\leqq J_i(g)\quad-\int_{R_{\alpha,\beta}}g, \tag{1}$$

$$0\leqq J_i(f_{n,m})-\int_{R_{\alpha,\beta}}f_{n,m}\leqq J_i(g_{n,m})-\int_{R_{\alpha,\beta}}g_{n,m}. \tag{2}$$

For each $\varepsilon>0$ there exists a number $n_\epsilon$ so that, by hypothesis and by Lemma 3, for all values of $n$, $m>n_\epsilon$ and all values of $\alpha$, $\beta$ uniformly,

$$|J_i(g_{n,m})-J_i(g)|,\quad\left|\int_{R_{\alpha,\beta}}g-\int_{R_{\alpha,\beta}}g_{n,m}\right|<\varepsilon, \tag{3}$$

and so

$$\left|\left(J_i(g_{n,m})-\int_{R_{\alpha,\beta}}g_{n,m}\right)-\left(J_i(g)-\int_{R_{\alpha,\beta}}g\right)\right|<2\varepsilon. \tag{4}$$

But, if $\alpha$, $\beta>$ some number $\alpha_\epsilon$,

$$\left|J_i(g)-\int_{R_{\alpha,\beta}}g\right|<\varepsilon, \tag{5}$$

and, therefore, by (3) and (5),

$$\left|J_i(g_{n,m})-\int_{R_{\alpha,\beta}}g_{n,m}\right|<3\varepsilon,\qquad(n,\ m>n_\epsilon;\alpha,\ \beta>\alpha_\epsilon).$$

This, with (1) and (2), shows that

$$\left|J_i(f)-J_i(f_{n,m})+\int_{R_{\alpha,\beta}}f_{n,m}-\int_{R_{\alpha,\beta}}f\right|<4\varepsilon(n,\ m>n_\epsilon;\alpha,\ \beta>\alpha_\epsilon). \tag{6}$$

Now let $\alpha$, $\beta$ be held fast at some values greater than $\alpha_\epsilon$. There exists a number $n'_\epsilon$ so that

$$\left| \int_{R_{\alpha,\beta}} f_{n,m} - \int_{R_{\alpha,\beta}} f \right| < \varepsilon, \qquad (n, m > n'_\epsilon), \tag{7}$$

for, by Lemma 3, and since $0 \leqq f_{n,m} \leqq g_{n,m}$, Theorem 9 applies. Thus (6) and (7) prove the lemma.

THEOREM 12. *Let* $lim_{n,m} f_{n,m}(x, y) = f(x, y)$ *in general. In order that the following integrals shall exist and that*

$$lim_{n,m} J_i(f_{n,m}) = J_i(f),$$

*it is necessary and sufficient that there shall exist two convergent sequences of functions,* $lim_{n,m} g_{n,m}(x, y) = g(x, y)$, $lim_{n,m} G_{n,m}(x, y) = G(x, y)$, *for which this is true, such that in general* $g_{n,m}(x, y) < f_{n,m}(x, y) < G_{n,m}(x, y)$.

To prove the necessary part we may take $g_{n,m} = f_{n,m} - 1/(mnx^2y^2)$, $G_{n,m} = f_{n,m} + 1/(mnx^2y^2)$. To prove the sufficient part we note that $0 \leqq f_{n,m} - g_{n,m} \leqq G_{n,m} - g_{n,m}$, and that $lim_{n,m} J_i(G_{n,m} - g_{n,m}) = J_i(G - g)$, and apply Lemma 4.

COROLLARY 1. *Let* $lim_{n,m} f_{n,m}(x, y) = f(x, y)$ *in general. Any one of the following is sufficient to make* $lim_{n,m} J_i(f_{n,m}) = J_i(f)$:

(*a*) $g(x, y) \leqq f_{n,m}(x, y) \leqq G(x, y)$, *and* $J_i(g)$, $J_i(G)$ *shall exist;*

(*b*) $f(x, y)$ *shall be the limit of a monotonic sequence,*

$$f_1(x, y) \leqq f_2(x, y) \leqq \ldots,$$

*and* $J_i(f)$ *and* $J_i(f_1)$ *shall exist;*

(*c*) $lim_{n,m} J_i(|f_{n,m}|) = J_i(|f|)$;

(*d*) $|f_{n,m}(x, y)| \leqq G(x, y)$ *and* $J_i(G)$ *shall exist;*

(*e*) $f_{n,m}(x, y) = h_{n,m}(x, y) g_{n,m}(x, y)$, *where* $0 \leqq h_{n,m}$, $g_{n,m} \leqq 1$, *and one of the convergent sequences* $h_{n,m}$ *and* $g_{n,m}$ *shall be integrable termwise in* $R$;

(*f*) $f_{n,m} = (g_{n,m})^p$, $0 \leqq g_{n,m} \leqq 1 < p$, *and the sequence* $g_{n,m}$ *shall be integrable termwise in* $R$.

COROLLARY 2.* *The double series of which the general term is* $u_{n,m}(x, y)$ *is absolutely convergent and absolutely integrable termwise in* $R$ *if there exists a never negative function* $v(x, y)$ *which is integrable in* $R$ *and a convergent double series of positive terms of the type* $a_{n,m}$ *such that* $|u_{n,m}(x, y)| \leqq a_{n,m} v(x, y)$.

For, if $s_{n,m}$ refer to the sum of the absolute values of the $nm$ terms of the $u$ series, and $\sigma_{n,m}$ to the sum of the corresponding terms of the $a$ series, $0 \leqq s_{n,m} \leqq \sigma_{n,m} v(x, y) \leqq \lim_{n,m} \sigma_{n,m} v(x, y)$, and (*d*) of Corollary 1 applies.

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* Cf. W. H. Young, *loc. cit.*, p. 324. The same statement holds good if $R$ is a finite field.

8. **Differentiation and Integration.** The next theorem may be derived from what precedes in much the same manner as may be proved the theorem* of which it is a generalization.

THEOREM 13. *Let* $g_n(x, y) \leqq f_n(x, y) \leqq G_n(x, y)$, *and* $f_n - g_n$, $G_n - f_n$ *be monotonic increasing functions of* $n$ *at each point* $(x, y)$, *and let the derivative of* $J_i(g_n) = J_i(dg_n/dn)$, *and the derivative of* $J_i(G_n) = J_i(dG_n/dn)$. *Then the derivative with respect to* $n$ *of* $J_i(f_n) = J_i(df_n/dn)$ *wherever*† $df_n/dn$ *exists. The same is true if the right (left) hand derivative be used throughout instead of the derivative.*

The following lemma may be proved by the use of the theorem of Young analogous to Theorem 9, and the theorem of Fubini analogous to the corollary of Theorem 5.

LEMMA. *Let the interval* $(a \leqq n \leqq b)$ *be finite. Sufficient conditions that the following exist and be equal,*

$$\int_a^b J_1[g(x, y; n)]\,dn, \quad J_1\left[\int_a^b g(x, y; n)\,dn\right],$$

*are that* (1°) *the integral of* $g$ *with respect to* $x$ *and* $y$ *over* $R_{\alpha,\beta}$, *for all values of* $\alpha, \beta > \alpha_0$, *be absolutely less than an integrable function of* $n$, (2°) $J_1(g)$ *exist for each* $n$, *and* $g$ *be absolutely integrable with respect to* $x$, $y$, *and* $n$ *in the parallelopiped,* $0 \leqq x \leqq \alpha$, $0 \leqq y \leqq \beta$, $a \leqq n \leqq b$.

THEOREM 14. *Sufficient conditions that the equation,*

$$\int_a^n J_1\left(\frac{df}{dn}\right) dn = J_1(f_n) - J_1(f_a),$$

*be valid in* $(a \leqq n \leqq b)$ *are that* $J_1(f)$ *exist at one point of the interval* $(a, b)$, *and that* $df/dn$ *exist and satisfy the conditions imposed on* $g$ *in the lemma.*

By the lemma

$$\int_a^n J_1\left(\frac{df}{dn}\right) dn = J_1 \int_a^n \frac{df}{dn}\,dn.$$

By a known theorem due to Vallée-Poussin,‡ this equals $J_1(f_n) - J_1(f_a)$.

COROLLARY.§ *Under the same conditions the derivative, with respect to* $n$, *of* $J_1(f_n)$ *equals* $J_1$ *of the derivative of* $f_n$, (*a*) *in general in* $(a, b)$ *including all*

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* See preceding note, same reference, p. 321.

† More generally, for each value of $n$ for which, except perhaps for a null set in the $xy$-plane, $df/dn$ exists.

‡ *Loc. cit.*, Vol. I, p. 263.

§ See reference of previous note, pp. 267, 268.

*points where the derivative of $J_1(f_n)$ exists, and (b) at all points of $(a, b)$ provided $J_1(df/dn)$ is continuous in $(a, b)$.*

THE FOURIER INTEGRALS, §§ 9–11.

9. **Introduction.** The iterated integral

$$J_4=\frac{1}{\pi}\int_0^\infty du\int_{-\infty}^\infty f(t)\cos u(t-x)\,dt$$

is sometimes called the Fourier double integral. We shall reserve this name for

$$J_1=D=\lim_{\alpha,\,\beta=\infty}\frac{1}{\pi}\int_0^\alpha du\int_0^\beta f(t)\cos u(t-x)\,dt+\lim_{\alpha,\,\beta=\infty}\frac{1}{\pi}\int_0^\alpha du\int_{-\beta}^0 f(t)\cos u(t-x)\,dt.$$

Here $J_1$ and $J_4$ have the same relative significance as in the preceding sections. It is evident that $J_5$ does not exist. A good deal has been written * recently on the problem of finding the most general conditions that one can impose on $f(t)$ and make $J_4$ equal to $f(x)$. In the following section, it is shown that $J_1$ equals $f(x)$ under circumstances more general than those derived for $J_4$. Indeed, if $f(x)=1$, $J_4$ does not exist, and if a value be given it by the usual summation method, that value is zero, but $J_1=1$. In Section 11 the corresponding integral which arises from three-dimensional problems is considered, and some of the questions connected with it. For simplicity $x$, $y$, and $z$ are restricted to non-negative values. This enables one to dispense with the second term of $D$. No generality is lost thereby.

10. **The Fourier Double Integral.** LEMMA. *(a) For all $\alpha, \beta \geqq 0$ uniformly,*

$$\left|\int_0^\alpha \frac{\sin\beta t}{t}\,dt\right| \leqq \frac{\pi}{2}.$$

*(b) Let $0\leqq f(t+x)\leqq M$, and let $f$ be a monotonic decreasing function of $t\,(t\geqq 0)$ for each $x$. Then, uniformly with respect to $\alpha\geqq\alpha_0>0$, and boundedly † with respect to $x$ in the first case, and uniformly in the second:*

$$\lim_{\beta=\infty}\int_0^\alpha f(t+x)\,\frac{\sin\beta t}{t}\,dt=\frac{\pi}{2}f(+0+x),\quad \lim_{\beta=\infty}\int_{\alpha_0}^\alpha f(t+x)\,\frac{\sin\beta t}{t}\,dt=0.$$

*(c) If also uniformly with respect to $x$ $\lim\limits_{t=0} f(t+x)=f(+0+x)$, the convergence in the first case under (b) is uniform.*

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* A. Pringsheim, *Mathematische Annalen,* Vol. LXVIII (1910), pp. 367–408. W. H. Young, *Transactions of the Cambridge Philosophical Society,* Vol. XXI (1910), pp. 428–451; *Proceedings of the Royal Society of Edinburgh,* Vol. XXXI (1911), pp. 559–586.

† $F_n(x, y, \ldots.)$ approaches its limit "boundedly" with respect to $x, y, \ldots.$ if there exists a fixed number greater than $|F_n|$ uniformly for all values of $x, y, \ldots.$.

The lemma is known except perhaps for the boundedness and uniformity of the convergence. By the more familiar part, however, we have

$$\frac{\pi}{2} f(+0+x) - \lim_{\beta=\infty} \int_0^a f(t+x) \frac{\sin \beta t}{t} dt$$
$$= \lim_{\beta=\infty} \int_0^a [f(+0+x) - f(t+x)] \frac{\sin \beta t}{t} dt = \lim_{\beta=\infty} S.$$

By $(a)$ and the second theorem of the mean,

$$|S| \leqq M \max_{a} \left| \int_0^a \frac{\sin \beta t}{t} \right| \leqq \frac{\pi M}{2}.$$

Thus the convergence is bounded. In the cases where it is uniform the demonstration is the same except that $M$ is to be chosen arbitrarily.

THEOREM 15. *Let $0 \leqq f(x) \leqq M$, and let $f(x)$ be a monotonic decreasing function of $x (x \geqq 0)$. Then $D = 1/2 [f(x-0) + f(x+0)]$, and the proper integral of which $D$ is the limit approaches $D$ boundedly with respect to $x$ if $x \geqq a > 0$.*

CASE 1: $(x=0)$. By the lemma

$$\pi D = \lim_{a,\,\beta=\infty} \int_0^a du \int_0^\beta f(t) \cos tu \, dt = \lim_{a,\,\beta=\infty} \int_0^\beta f(t) \frac{\sin at}{t} dt = \frac{\pi}{2} f(+0).$$

CASE 2: $(0 < a \leqq x)$. In the formal expression for $D$ substitute $t - x = s$. Then

$$\pi D = \lim_{a,\,\beta=\infty} \int_0^a du \int_{-x}^{\beta - x} f(s+x) \cos su \, ds = \lim_{a,\,\beta=\infty} \int_{-x}^{\beta-x} f(s+x) \frac{\sin \alpha s}{s} ds$$
$$= \lim_{a,\,\beta=\infty} \int_{-x}^{0} f(s+x) \frac{\sin \alpha s}{s} ds + \lim_{a,\,\beta=\infty} \int_0^{\beta-x} f(s+x) \frac{\sin \alpha s}{s} ds.$$

Substituting $t = -s$ in the first integral, and separating the last integral into two parts by the point $x$,

$$\pi D = \lim_{a,\,\beta=\infty} \int_0^x [f(-t+x) + f(t+x)] \frac{\sin \alpha t}{t} dt + \lim_{a,\,\beta=\infty} \int_x^{\beta-x} f(s+x) \frac{\sin \alpha s}{s} ds,$$

which equals $\frac{1}{2}[f(x-0) + f(x+0)]$ by $(b)$ of the lemma, and the convergence is bounded.

COROLLARY 1. *If also $f(x)$ is continuous in $(0 < a \leqq x \leqq b)$, the convergence is uniform there.*

Apply $(c)$ of the lemma.

COROLLARY 2. *The theorem and corollary are true if $f(x) = f_1(x) - f_2(x)$, where $f_1(x)$ and $f_2(x)$ have the properties there ascribed to $f(x)$.*

*Examples:* $e^{-x}$, $1 - e^{-x}$, $f(x)$ equal to a polynomial in $(0, 1)$ and to $1/x$ elsewhere.

THEOREM 16. *Let* (1°) $f(x)=g(x)p(x)$, *where* $0\leqq f, g, p\leqq M, f(x)$ *has limited variation in every finite** *interval, and* $g(x)$ *is monotonic decreasing* $(x\geqq x_0)$, *and* $p(x)$ *is periodic* $(x\geqq x_0)$. *Let* (2°), *except perhaps a null set, for* $x\geqq x_0$,

$$p(x)=\sum_{i=0}^{\infty}(a_i \sin kix+b_i \cos kix),$$

*where the* $a_i$*'s and* $b_i$*'s are the usual Fourier coefficients, and* (3°) $\Sigma(|a_i|+|b_i|)=H$. Then for each $x\geqq 0$, $D=1/2[f(x-0)+f(x+0)]$.

Let $f(t)=\phi(t)+\psi(t)$, where $\phi=f$ and $\psi=0$ at points where $t\leqq 2x$, and $\phi=0$ and $\psi=f$ where $t>2x$, $x$ being the point at which the development is to be examined. Referring to the preceding corollary we see that it holds when $\phi$ replaces $f$. Referring to the proof of the previous theorem we see that, by formal processes,

$$\pi D-\pi/2[f(x-0)+f(x+0)]=\lim_{\alpha,\beta=\infty}\int_0^x[\psi(-t+x)+\psi(t+x)]\frac{\sin \alpha t}{t}dt$$
$$+\lim_{\alpha,\beta=\infty}\int_x^{\beta-x}\psi(s+x)\frac{\sin \alpha s}{s}ds=\mathrm{I}+\mathrm{II}.$$

By definition of $\psi$, $\mathrm{I}=0$. To prove that $\mathrm{II}=0$ we have essentially to establish for the $f$ of our present theorem only the second part of $(b)$ of the lemma to Theorem 15, with the omission of the references to boundedness and uniformity with respect to $x$. An apparent exception occurs when $x=0$, but it may be avoided by separating the interval $(\alpha,\beta)$ into two parts by means of a fixed point. We proceed, therefore, to show that, $\alpha_0>0$, $\varepsilon>0$, $x$ being prescribed, there exists a $\beta_\varepsilon$ so that the following is true uniformly in $\alpha$ and $\beta$:

$$\left|\int_{\alpha_0}^{\alpha}g(t+x)p(t+x)\frac{\sin \beta t}{t}dt\right|<\varepsilon, \qquad (\beta>\beta_\varepsilon).$$

The function $g(t+x)/t$ is monotonic decreasing if $t>x_0$. Let $\alpha_1$ be so large that $\alpha_1>x_0$ and also that $g(t+x)/\alpha_1<\varepsilon$. Since† $gp/t$ is absolutely $L$-integrable in $(\alpha_0, \alpha_1)$,

$$\lim_{\beta=\infty}\int_{\alpha_0}^{\alpha_1}\frac{gp}{t}\sin \beta t dt=0.$$

Now

$$\left|\int_{\alpha_1}^{\alpha}\frac{g(t+x)}{t}p(t+x)\sin \beta t dt\right|<\varepsilon \max_{\lambda}\left|\int_{\alpha_1}^{\lambda}p(t+x)\sin \beta t dt\right|$$
$$=\varepsilon \max_{\lambda}\left|\int_{\alpha_1}^{\lambda}\sin \beta t\left[\sum_0^{\infty}a_i \sin ki(t+x)+b_i \cos ki(t+x)\right]dt\right|.$$

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*These conditions could be generalized a little by requiring that certain of them should hold necessarily only in the neighborhood of $x$.

†Lebesgue, *Annales de la Faculté de Toulouse*, Ser. 3, Vol. I (1909), p. 52.

It may be seen in several ways that this series is integrable termwise; for example, by 3° its remainder is uniformly limited. Thus the above expression equals

$$\varepsilon \max_{\lambda} \left| \sum_{0}^{\infty} \left[ a_i \int_{a_1}^{\lambda} \sin ki(t+x) \sin \beta t dt \right.\right.$$
$$\left.\left. + b_i \int_{a_1}^{\lambda} \cos ki(t+x) \sin \beta t dt \right] \right| \leqq 8\varepsilon \Sigma (|a_i| + |b_i|) = 8\varepsilon H.$$

Corollary. *The theorem is true if* $f(x) = f_1(x) - f_2(x)$, *where* $f_1$ *and* $f_2$ *separately enjoy the properties there ascribed to* $f$.

*Example:* $f = x$ in $(0, \pi)$, $= 2\pi - x$ in $(\pi, 2\pi)$, etc.

Theorem 17. *If* $f(x)$ *has limited variation in a neighborhood* $(x-c, x+c)$, $c > 0$, *of* $x$, *and if* $f(x)/x$ *is absolutely L-integrable in the infinite interval* $(x \geqq c)$, *then, for each* $x \geqq 0$, $D = 1/2[f(x-0) + f(x+0)]$.

As before

$$D = \lim_{\alpha, \beta = \infty} \frac{1}{\pi} \int_{-x}^{\beta - x} f(t+x) \frac{\sin \alpha t}{t} dt = \lim_{\alpha, \beta = \infty} \frac{1}{\pi} \left[ \int_{-x}^{-\frac{c}{2}} \frac{f \sin \alpha t}{t} dt \right.$$
$$\left. + \int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{f \sin \alpha t}{t} dt + \int_{\frac{c}{2}}^{\beta - x} \frac{f \sin \alpha t}{t} dt \right] = \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

$$\mathrm{II} = \lim_{\alpha, \beta = \infty} \frac{1}{\pi} \int_{0}^{\frac{c}{2}} \tfrac{1}{2} [f(-t+x) + f(t+x)] \frac{\sin \alpha t}{t} dt = \tfrac{1}{2} [f(x-0) + f(x+0)],$$

since the part of the integrand in brackets has limited variation in $(0, c/2)$. Since $f(t+x)/t$ is absolutely $L$-integrable in $(-x, -c/2)$, by Lebesgue's* theorem $\mathrm{I} = 0$; and, similarly, a part of III,

$$\lim_{\alpha, \beta = \infty} \frac{1}{\pi} \int_{\frac{c}{2}}^{\beta_\varepsilon} f(t+x) \frac{\sin \alpha t}{t} dt = 0,$$

where $\beta_\varepsilon$ is so chosen that

$$\int_{\beta_\varepsilon}^{\infty} \left| \frac{f(t+x)}{t} \right| dt < \varepsilon.$$

Thus $|\mathrm{III}| < \varepsilon$.

*Example:* $f(x) = p(x)e^{-x}$, where $p$ is any limited function which has limited variation in the neighborhood of $x$ and is $L$-integrable in every finite interval.

It could now be shown that, if in this form of the Fourier integral the integrand were multiplied by $e^{-uy}$ (or by $e^{-c^2u^2t}$), the usual thermostatic (or thermodynamic) equations would be satisfied, in general, and that the functions

* *Loc. cit.*

defined thereby would enjoy the other properties desired for the solutions of the corresponding problems in heat and electricity. But this would not be useful, for, although some texts imply that the Fourier "double" (in my sense, iterated) integral is needed in these physical problems, it never is, and the appropriate propositions can be proved for the simple integrals that are needed in a much more general fashion than I can prove them even for the double integrals considered here.* The theorems of this section serve as lemmas to the next section, where their practical value becomes apparent. However, as notably in the case of Fourier's series,† so to a less degree in the case of Fourier's integral, the theory of the representation of an arbitrary function has enjoyed a development and interest that have nothing to do with its value as a physical tool.

11. **The Fourier Quadruple Integral.** If one seeks the value of the potential function $V$ in an infinite rectangular parallelopiped $(x, y, z \geqq 0)$, given that the surface $z=0$ is kept at a potential $V_0=f(x, y)$, and that the surface $x=0$ is insulated, one is led to LaPlace's equation, $\nabla^2(V)=0$, and to the formula: ‡

$$V_z = \lim_{\alpha, \beta, \gamma, \delta=\infty} \int_0^\alpha dt \int_0^\beta du \int_0^\gamma ds \int_0^\delta e^{-z\sqrt{s^2+w^2}} f(t, u) \cos s(t-x) \cos w(u-y)\, dw.$$

THEOREM 18. *Let $f(x, y)=h(x)k(y)$, where $h$ and $k$ satisfy the conditions imposed on $f$ in one of the theorems* 15, 16, *or* 17, *or are the differences of two such functions. Then $V_0=1/4[f(x-0, y-0)+f(x-0, y+0)+f(x+0, y-0)+f(x+0, y+0)]$.*

For $V_0 = \lim_{\alpha, \beta, \gamma, \delta} \left[\int_0^\alpha dt \int_0^\gamma h(t) \cos s(t-x)\, ds\right]\left[\int_0^\beta du \int_0^\delta k(y) \cos w(u-y)\, dw\right]$,

which equals $1/4[h(x-0)+h(x+0)][k(y-0)+k(y+0)]$.

COROLLARY. *The convergence is uniform with respect to $x$ (or $y$) if $h$ (or $k$) also satisfies the condition of continuity imposed in the corollary to Theorem* 15.

THEOREM 19. *If $f(x, y)$ is absolutely L-integrable in the infinite rectangle $(x, y \geqq 0)$, $V_z(z>0)$ satisfies LaPlace's equation.*

It is sufficient to show that the signs of partial differentiation may be passed over the limit sign and over the signs of integration in the expression for $V_z$. For this purpose we refer to corollary ($b$) of Theorem 14. Using

* Cf. W. H. Young, *Proceedings of the Royal Society of Edinburgh*, Vol. XXXI (1911), p. 587; *Encyklopädie Mathematischen Wissenschaften*, II, A 12, p. 1089.

† Cf. VanVleck, "The Influence of Fourier's Series upon the Development of Mathematics," *Science*, N. S., Vol. XXXIX (1914), Number 995, pp. 113–124.

‡ In the usual formula the iterated integral is used.

the notation there, but replacing that $f$ by the present integrand, which we call $\phi$, and thinking of $n$ as representing in turn $x$, $y$, and $z$, we see that we need to prove: (1) that $J_1(d\phi/dn)$ is continuous near $n$, $n$ being the point at which the derivative of $J_1$ is being examined; (2) that $J_1(\phi)$ exists at $n$; (3) that $d\phi/dn$ exists, and is absolutely $L$-integrable over finite three-dimensional fields; and (4) that there exists a function, $\Phi(n)$, whose integral with respect to $n$ exists over a finite interval enclosing the particular point $n$ in question, such that, for all values of $\alpha, \beta, \gamma, \delta \geqq 0$ uniformly,

$$\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \frac{d\phi}{dn}\, dt\, du\, ds\, dw \right| < \Phi(n).$$

These results are needed in order to show that the first derivative of $J_1$ may be replaced by the first derivative of its integrand. The process must be repeated for the second derivative. We shall then have conditions (5), (6), (7), and (8) exactly like (1), (2), (3), and (4), respectively, except that $d\phi/dn$ replaces $\phi$, and in (4) $\Phi(n)$ is replaced by some other function having the same properties. It is obvious that (3) and (7) are true, and that (6) is included in (1). We first prove (4). For all values of $x$, $y$, $z$ uniformly,

$$|\phi| \leqq |f(t,u)|\, e^{-z\sqrt{s^2+w^2}} \leqq |f|\, e^{-z\left(\frac{s+w}{2}\right)} = |f|\, e^{-\frac{zs}{2}}\, e^{-\frac{zw}{2}},$$

$$\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \phi \right| \leqq M \left( \int_0^\infty e^{-\frac{zs}{2}}\, ds \right)^2 = \frac{4M}{z^2},$$

where

$$M = \lim_{\alpha,\beta} \int_0^\alpha dt \int_0^\beta |f(t,u)|\, du.$$

Thus (2) $J_1(\phi)$ exists absolutely. Now, first let $n = z$.

$$\left.\begin{aligned} &\frac{\partial\phi}{\partial z} = -\sqrt{s^2+w^2}\,\phi, \quad \frac{\partial^2\phi}{\partial z^2} = (s^2+w^2)\phi, \quad \left|\frac{\partial\phi}{\partial z}\right| \leqq |f|\,(s+w)\,e^{-\frac{zs}{2}}\, e^{-\frac{zw}{2}}, \\ &\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \frac{\partial\phi}{\partial z} \right| \leqq 2M \left( \int_0^\infty s e^{-\frac{zs}{2}}\, ds \int_0^\infty e^{-\frac{zw}{2}}\, dw \right) = \frac{16M}{z^3}. \end{aligned}\right\} \quad (9)$$

Thus (4) is satisfied ($z > 0$). We can similarly prove (8), for

$$\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \frac{\partial^2\phi}{\partial z^2} \right| \leqq 2M \int_0^\infty s^2 e^{-\frac{zs}{2}}\, ds \int_0^\infty e^{-\frac{zw}{2}}\, dw = \frac{64M}{z^4}.$$

Now, by referring to Corollary 1 ($d^\circ$) of Theorem 12, it will be seen that in (9) we have established (1) and (5); *e. g.*, if $z \geqq c > 0$,

$$|f|\,(s+w)\,e^{-\frac{z}{2}(s+w)} \leqq |f|\,(s+w)\,e^{-\frac{c}{2}(s+w)},$$

which is an integrable function in $R$. Secondly, let $n = x \geqq 0$.

$$\frac{\partial \phi}{\partial x} = se^{-z\sqrt{s^2+w^2}} f(t, u) \sin s(t-x) \cos w(u-y),$$

$$\frac{\partial^2 \phi}{\partial x^2} = -s^2 e^{-z\sqrt{s^2+w^2}} \cos s(t-x) \cos w(u-y).$$

$$\left| \int_{R_{\alpha, \beta, \gamma, \delta}} \frac{\partial \phi}{\partial x} \right| < M \int_0^\infty s e^{-\frac{zs}{2}} ds \int_0^\infty e^{-\frac{zw}{2}} dw = \frac{8M}{z^3}.$$

$$\left| \int_{R_{\alpha, \beta, \gamma, \delta}} \frac{\partial^2 \phi}{\partial x^2} \right| < M \int_0^\infty s^2 e^{-\frac{zs}{2}} ds \int_0^\infty e^{-\frac{zw}{2}} dw = \frac{32M}{z^4}.$$

Hence the conditions in question are satisfied. The case, $n = y \geqq 0$, is identical with this one.

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